JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 6, November–December 1993

Flight Control Application of New Stability Robustness Bounds for Linear Uncertain Systems

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This paper addresses the issue of obtaining bounds on the real parameter perturbations of a linear state-space model for robust stability. Based on Kronecker algebra, new, easily computable sufficient bounds are derived that are much less conservative than the existing bounds since the technique is meant for only real parameter perturbations (in contrast to specializing complex variation case to real parameter case). The proposed theory is illustrated with application to several flight control examples.

I. Introduction

RECENTLY, the aspect of developing explicit upper bounds on the perturbation of linear state-space systems to maintain stability has received much attention; for example, see Yedavalli.[1] Out of the different methods available for structured uncertainty (which is of concern in this paper), the papers by Yedavalli,[2] Zhou and Khargonekar,[3] Juang et al.,[4] and Qiu and Davison[5] present sufficient bounds for robust stability, but these bounds are quite conservative. Necessary and sufficient conditions for stability robustness of linear state-space models with real parameter uncertainty are derived by Tesi and Vicino,[6] and it is now well known that this problem essentially involves the testing of positivity of a multivariate polynomial in real variables that becomes computationally intensive for more than two to three parameters. An explicit necessary and sufficient bound is presented by Fu and Barmish,[7] but for only a single uncertain parameter. Thus, the aspect of obtaining less conservative sufficient bounds for a large number of uncertain parameters is still an important issue of interest, especially for use in applications. With this in mind, in this paper, we present such sufficient bounds. This is accomplished using some Kronecker related matrices (which, of course, also were the tool in Refs. 6 and 7). The reduction in conservatism of the proposed method is due to the fact that this method distinguishes real parameter variations from complex parameter variations in the derivation of the sufficient condition. Similar treatment for unstructured uncertainty is given in Qiu and Davison.[8] It should be kept in mind that the bounds for structured uncertainty presented in this paper are considerably different from and improved over the bounds one can derive for structured uncertainty from the bounds for unstructured uncertainty.

The paper is organized as follows. In Sec. II, we briefly review the nominal matrix stability conditions from Fuller.[9] Section III extends these concepts to uncertain matrices and presents bounds on the perturbations for robust stability. Section IV illustrates the theory with some examples from flight control applications, and finally Sec. V offers some concluding remarks.

II. Stability Conditions for a Nominal Matrix

Before presenting some new results for uncertain systems, in this section, we briefly review a few stability theorems for a nominal system matrix A, in terms of Kronecker related matrices. Most of the following material is adopted from Fuller.[9]

Received July 27, 1992; revision received March 5, 1993; accepted for publication March 8, 1993.

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Definition 1: Let A be an n-dimensional matrix $[a_{ij}]$ and B an m-dimensional matrix $[b_{ij}]$. The mn-dimensional matrix C defined by

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ a_{21}B & \cdots & a_{2n}B \\ \vdots & & \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \tag{1}$$

is called the Kronecker product of A and B and is written

$$A \times B = C \tag{2}$$

Theorem 1: Let the characteristic roots of matrices A and B be $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_m$, respectively. Then the characteristic roots of the matrix

$$\sum_{p,q} h_{pq} A^p \times B^q \tag{3}$$

are the mn values $\Sigma_{p,q} h_{pq} \lambda_i^p \mu_j^q$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Corollary 1: The characteristic roots of the matrix $A \oplus B$ where

$$A \oplus B = A \times I_m + I_n \times B \tag{4}$$

are the mn values $\lambda_i + \mu_j$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The matrix $A \oplus B$ is called the Kronecker sum of A and B.

Kronecker Sum of A with Itself: Let $\mathfrak{D}$ be the matrix of dimension $k = n^2$, defined by

$$\mathfrak{D} = A \times I_n + I_n \times A \tag{5}$$

Corollary 2: The characteristic roots of $\mathfrak{D}$ are $\lambda_i + \lambda_j$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Henceforth, we use an operator notation to denote $\mathfrak{D}$. We write $\mathfrak{D} = K[A]$.

Example 1: For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with λ_1 and λ_2 as eigenvalues, the previous $\mathfrak{D}$ matrix is given by

$$\mathfrak{D} = \begin{bmatrix} 2a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{11}+a_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22}+a_{11} & a_{12} \\ 0 & a_{21} & a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $2\lambda_1$, $\lambda_1 + \lambda_2$, $\lambda_2 + \lambda_1$, and $2\lambda_2$.

Stability Condition I (for Nominal Matrix A in Terms of Kronecker Sum Matrix $\mathfrak{D}=K[A]$)

Theorem 2: For the characteristic roots of A to have all of their real parts negative (i.e., for A to be asymptotically stable), it is necessary and sufficient that in the characteristic polynomial

$$(-1)^k|K[A]-\lambda I_k| \tag{6}$$

the coefficients of λ_i ($i=0, 1, 2, \ldots, k-1$) should all be positive.

We now define another Kronecker related matrix $\mathfrak{L}$ called "Lyapunov matrix" and state a stability theorem in terms of this matrix.

Definition 2—Lyapunov Matrix $\mathfrak{L}$: The elements of the Lyapunov matrix $\mathfrak{L}$ of dimension $\ell=(½)[n(n+1)]$ in terms of the elements of the matrix A are given as follows:

For $p>q$,

$$\mathfrak{L}_{pq,rs} = \left|\begin{array}{lll} a_{ps} & \text{if} & r=q \text{ and } s<q \\ a_{pr} & \text{if} & r\geq q,\ r\neq p \text{ and } s=q \\ a_{pp}+a_{qq} & \text{if} & r=p \text{ and } s=q \\ a_{qs} & \text{if} & r=p \text{ and } s\leq p,\ s\neq q \\ a_{qr} & \text{if} & r>p \text{ and } s=p \\ 0 & & \text{otherwise} \end{array}\right| \tag{7}$$

and for $p=q$,

$$\mathfrak{L}_{pq,rs} = \left|\begin{array}{lll} 2a_{ps} & \text{if} & r=p \text{ and } s<p \\ 2a_{pp} & \text{if} & r=p \text{ and } s=p \\ 2a_{pr} & \text{if} & r>p \text{ and } s=p \\ 0 & & \text{otherwise} \end{array}\right| \tag{8}$$

Corollary 3: The characteristic roots of $\mathfrak{L}$ are $\lambda_i+\lambda_j$ ($i=1, 2, \ldots, n; j=1, 2, \ldots, i$).

Example 2: If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{9}$$

with eigenvalues λ_1 and λ_2, then the Lyapunov matrix is given by

$$\mathfrak{L} = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11}+a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $2\lambda_1$, $\lambda_1+\lambda_2$, and $2\lambda_2$. We observe that, when compared with the eigenvalues of the Kronecker sum matrix $\mathfrak{D}$, the eigenvalues of $\mathfrak{L}$ omit the repetition of eigenvalues $\lambda_1+\lambda_2$. Again, for simplicity, we use operator notation to denote $\mathfrak{L}$. We write $\mathfrak{L}=L[A]$. A method to form the $\mathfrak{L}$ matrix from the matrix $\mathfrak{D}$ is given in Tesi and Vicino.[6]

Stability Condition II (for Nominal Matrix A in Terms of Lyapunov Matrix $\mathfrak{L}=L[A]$)

Theorem 3: For the characteristic roots of A to have all of their real parts negative (i.e., for A to be an asymptotically stable matrix), it is necessary and sufficient that in the characteristic polynomial

$$(-1)^\ell|L[A]-\lambda I_\ell| \tag{10}$$

the coefficients of λ_i ($i=0, 1, \ldots, \ell-1$) should all be positive.

Clearly, Theorem 3 is an improvement over Theorem 2, since the dimension of $\mathfrak{L}$ is less than that of $\mathfrak{D}$.

Finally, there is another matrix, called "bialternate sum" matrix, of reduced dimension $m=(½)[n(n-1)]$ in terms of which a stability theorem like that given earlier can be stated.

Definition 3—Bialternate Sum Matrix $\mathcal{G}$: The elements of the bialternate sum matrix $\mathcal{G}$ of dimension $m=(½)[n(n-1)]$ in terms of the elements of the matrix A are given as follows:

$$\mathcal{G} = \begin{bmatrix} -a_{ps} & \text{if} & r=q \text{ and } s<q \\ a_{pr} & \text{if} & r\neq p \text{ and } s=q \\ a_{pp}+a_{qq} & \text{if} & r=p \text{ and } s=q \\ a_{qs} & \text{if} & r=p \text{ and } s\neq q \\ -a_{qr} & \text{if} & s=p \\ 0 & & \text{otherwise} \end{bmatrix} \tag{11}$$

Note that $\mathcal{G}$ can be written as $\mathcal{G}=A\cdot I_n+I_n\cdot A$ where $\cdot$ denotes the bialternate product (see Jury[10] for details on the bialternate product). Again, we use operator notation to denote $\mathcal{G}$. We write $\mathcal{G}=G[A]$.

Corollary 4: The characteristic roots of $\mathcal{G}$ are $\lambda_i+\lambda_j$, for $i=2, 3, \ldots, n$ and $j=1, 2, \ldots, i-1$.

In Jury,[10] a simple computer-amenable methodology is given to form $\mathcal{G}$ matrix from the given matrix A.

Example 3: For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

with λ_1 and λ_2 as eigenvalues, the bialternate sum matrix $\mathcal{G}$ is given by the scalar

$$\mathcal{G} = [a_{22}+a_{11}]$$

where the characteristic root of $\mathcal{G}$ is $\lambda_1+\lambda_2=a_{11}+a_{22}$.

Example 4: When $n=3$, for the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with λ_1, λ_2, and λ_3 as eigenvalues, the bialternate sum matrix $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{bmatrix} a_{22}+a_{11} & a_{23} & -a_{13} \\ a_{12} & a_{33}+a_{11} & a_{12} \\ -a_{31} & a_{21} & a_{33}+a_{22} \end{bmatrix} \tag{12}$$

with eigenvalues $\lambda_1+\lambda_2$, $\lambda_2+\lambda_3$, and $\lambda_3+\lambda_1$.

Note that, when compared with the eigenvalues of $\mathfrak{D}$ and $\mathfrak{L}$, the eigenvalues of $\mathcal{G}$ omit the eigenvalues of the type $2\lambda_i$.

Stability Condition III (for Nominal Matrix A in Terms of the Bialternate Sum Matrix $\mathcal{G}$)

Theorem 4: For the characteristic roots of A to have all of their real parts negative, it is necessary and sufficient that in $(-1)^n$ times the characteristic polynomial of A, namely,

$$(-1)^n|A-\lambda I_n| \tag{13}$$

and in $(-1)^m$ times the characteristic polynomial of $\mathcal{G}$, namely,

$$(-1)^m|G[A]-\mu I_m| \tag{14}$$

the coefficients of λ^i ($i=0, \ldots, n-1$) and μ^i ($i=0, \ldots, m-1$) should all be positive.

This theorem improves somewhat on Theorems 2 and 3, since the dimension of $\mathcal{G}$ is less than the dimensions of $\mathfrak{D}$ and $\mathfrak{L}$, respectively.

One important consequence of the fact that the eigenvalues of $\mathfrak{D}$, $\mathfrak{L}$, and $\mathcal{G}$ include the sum of the eigenvalues of A is the following fact, which is stated as a lemma to emphasize its importance.

Lemma 1:

$$\det K[A] = 0$$

$$\det L[A] = 0$$

$$\det G[A] = 0$$

if and only if at least one complex pair of the eigenvalues of A is on the imaginary axis and

$$\det A = 0$$

if and only if at least one of the eigenvalues of A is at the origin of the complex plane.

It is important to note that $\det K[A]$, $\det L[A]$, and $\det G[A]$ represent the constant coefficients in the corresponding characteristic polynomials mentioned earlier. It may also be noted that the previous lemma explicitly takes into account the fact that the matrix A is a real matrix and hence has eigenvalues in complex conjugate pairs. This is the main reason for the robustness theorems based on these matrices, which are given in the next section, to give less conservative bounds as compared with other methods that do not distinguish between real and complex matrices.

III. New Perturbation Bounds for Robust Stability

In this section, we extend the concepts of stability of a nominal matrix in terms of Kronecker theory given in the previous section to perturbed matrices and derive bounds on the perturbation for robust stability for systems with structured uncertainty. Toward this direction, we consider a system with structured perturbation as follows:

$$\dot{x} = A(q)x \qquad x(0) = x_0 \tag{15}$$

where

$$A(q) = A_0 + \sum_{i=1}^{\ell} f_i(q-q^0)A_i = A_0 + E(q) \tag{16}$$

with $A_0 = A(q^0) \in R^{n\times n}$ being the "nominal" matrix, $A_i \in R^{n\times n}$ are given constant matrices, $f_i(\cdot)$ are scalar polynomial functions such that $f_i(0) = 0$, and the parameter vector $\underline{q}^T = [q_1, q_2, \ldots, q_r]$ belongs to the hyper-rectangular set $\Omega(\beta)$ defined by

$$\Omega(\beta) = (q \in R^r : q_i^0 - \beta \underline{w}_i \le q_i \le q_i^0 + \beta \bar{w}_i)$$

for $i = 1, 2, \ldots, r$, where $\beta > 0$ and $\underline{w}_i$ and $\bar{w}_i$, for $i = 1, 2, \ldots, r$, are positive weights.

A special case of this general description is of interest.

Linear Dependent Variations

For this case,

$$E(q) = \sum_{i=1}^{r} q_i A_i \tag{17}$$

where A_i are constant, given matrices with no restriction on the structure of the matrix A_i. This type of representation produces a "polytope of matrices" in the matrix space.

A special case of this is the so-called "independent variations" case, given by

$$E(q) = \sum_{i=1}^{r} q_i E_i \tag{18}$$

where E_i contains a single nonzero element at a different location in the matrix for different i. In this case, the set of possible $A(q)$ matrices forms a hyper-rectangle in $R^{n\times n}$. In this representation, the family of matrices is labeled the "interval matrix" family.

It may be noted that, even though only analysis is presented here, in a design situation, the matrix A_0 may represent the nominal closed-loop system matrix with gain matrix elements as design parameters.

In what follows, we extend the previous theorems to present necessary and sufficient conditions for robust stability of linear uncertain systems with structured uncertainty.

At this point, it is useful to mention that the operators $K[A]$, $L[A]$, and $G[A]$ satisfy the linearity property, namely,

$$K[A+B] = K[A] + K[B]$$

$$L[A+B] = L[A] + L[B]$$

$$G[A+B] = G[A] + G[B]$$

Theorem 5 (Robust Stability Condition Based on Kronecker Sum Matrix K [A]): The perturbed system (15) is stable if and only if

$$\det\left(\mathrm{I} + \left\{\sum_{i=1}^{\ell} f_i(q-q^0)K[A_i]\right\}\left(K[A_0]\right)^{-1}\right) > 0, \quad q \in \Omega(\beta) \tag{19}$$

The proof is given in Appendix A.

It may be noted that the results of Ref. 3 can be cast in the form of the Kronecker sum matrix $K[\cdot]$, and this is done in Hagood[11] where parametric Lyapunov equations are solved in terms of $K[\cdot]$ matrices. However, the sufficient bounds to be presented next are quite different and improved over the sufficient bounds of Zhou and Khargonekar[3] and Hagood.[11]

Theorem 6 (Robust Stability Condition Based on Lyapunov Matrix L[A]): The perturbed system (15) is stable if and only if

$$\det\left(\mathrm{I} + \left\{\sum_{i=1}^{\ell} f_i(q-q^0)L[A_i]\right\}\left(L[A_0]\right)^{-1}\right) > 0, \quad q \in \Omega(\beta) \tag{20}$$

The proof is given in Tesi and Vicino[6] and is along similar lines to the one in Appendix A.

Theorem 7 (Robust Stability Condition Based on Bialternate Sum Matrix G [A]): The perturbed system (15) is stable if and only if

$$\det\left\{I + \left[\sum_{i=1}^{\ell} f_i(q-q^0)A_i\right]A_0^{-1}\right\} > 0$$

and

$$\det\left(I + \left\{\sum_{i=1}^{\ell} f_i(q-q^0)G[A_i]\right\}\left(G[A_0]\right)^{-1}\right) > 0 \tag{21}$$

with $q \in \Omega(\beta)$.

The proof is similar to the proof given in Appendix A with appropriate modifications.

There is an interesting observation to be made from these theorems. Although for a nominal matrix A_0 to be stable, the necessary and sufficient condition is that all of the coefficients in the respective characteristic polynomials have to be positive, for a perturbed matrix $A_0 + E$, with A_0 being stable, the necessary and sufficient condition for stability requires the positivity of only the constant coefficient of the appropriate characteristic polynomial, which in turn is simply the determinant of the matrix being considered in the characteristic polynomial.

These previous theorems also imply that the robust stability problem can be converted to the positivity testing of multivariate polynomials over a hyper-rectangle in parameter space. This problem has been studied extensively in the literature.[10] The conclusion from this research is that this problem is computationally intensive and is extremely cumbersome to carry out when a large number of parameters are involved. The question of whether to go for necessary and sufficient bounds with huge computational effort for a small number of parameters or to settle for sufficient bounds with a relatively simpler computational effort but suitable for a large number of parameters is clearly dictated by the application at hand. In this research, the latter viewpoint is taken because in many applications such as aircraft control there are a large number of uncertain parameters present, and thus obtaining less conservative sufficient bounds for a large number of uncertain parameters is still of interest from the design point of view. Hence, in what follows, we derive sufficient conditions for robust stability of linear uncertain systems with structured uncertainty, applicable even when there are a large number of uncertain parameters.

Sufficient Bounds for Robust Stability

Theorem 8 (Based on Kronecker Sum Matrix $K[\cdot]$): The perturbed system (15) is stable if

$$\max_i \max_q |f_i(q-q^0)| < \mu_k \tag{22}$$

where

$$\mu_k = \frac{1}{\rho\left\{\sum_{i=1}^{\ell} \left|K[A_i]\left(K[A_0]\right)^{-1}\right|\right\}}$$

[No modulus sign is necessary in the denominator of Eq. (22) for $i=1$.]

In the previous expressions $\rho[\cdot]$ is the spectral radius of the matrix $[\cdot]$, and $|[\cdot]|$ denotes the matrix formed with the absolute values of the elements of $[\cdot]$.

The proof is given in Appendix B.

Theorem 9 (Based on Lyapunov Matrix $L[\cdot]$): The perturbed system (15) is stable if

$$\max_i \max_q |f_i(q-q^0)| < \mu_L \tag{23}$$

where

$$\mu_L = \frac{1}{\rho\left\{\sum_{i=1}^{\ell} \left|L[A_i]\left(L[A_0]\right)^{-1}\right|\right\}}$$

[No modulus sign is necessary in the denominator of Eq. (23) for $i=1$.]

The proof is very similar to the proof of Appendix B with $K[A_0]$ replaced by $L[A_0]$ and $K[A_i]$ replaced by $L[A_i]$.

Theorem 10 (Based on Bialternate Sum Matrix $G[\cdot]$): The perturbed system (15) is stable if

$$\max_i \max_q |f_i(q-q^0)| < \mu_G \tag{24}$$

where

$$\mu_G = \min\left(\mu_{A_0}, \mu_{A_G}\right)$$

$$\mu_{A_0} = \frac{1}{\rho\left(\sum_{i=1}^{\ell} |A_i A_0^{-1}|\right)}$$

$$\mu_{A_G} = \frac{1}{\rho\left\{\sum_{i=1}^{\ell} \left|G[A_i]\left(G[A_0]\right)^{-1}\right|\right\}}$$

[No modulus sign is necessary in Eq. (24) for $i=1$.]

The proof is similar to the proof in Appendix B with appropriate modifications.

IV. Application to Flight Control Problems

The aspect of stability robustness bounds has direct relevance to the application of aircraft control design. In a typical aircraft control design exercise, the aircraft dynamics are linearized about a given flight condition, and a control law is designed based on this linearized model. However, there is always uncertainty associated with the parameters of this control design model; for example, the aerodynamic stability derivatives are never known exactly, and there are always imperfections in actuator and sensor locations. Thus the control design for stabilization (and performance) should be able to accommodate these uncertainties in the model and to preserve stability (and performance) even in the presence of these uncertainties. In other words, the control design must possess the property of stability robustness. With this in mind, we present two examples in which the stability robustness of the given controllers is established by using the proposed bounds.

Application to VTOL Aircraft Control

The linearized model of the VTOL aircraft in the vertical plane is described by

$$\dot{x}(t) = \left[A_0 + \Delta A(t)\right]x(t) + \left[B_0 + \Delta B(t)\right]u(t) \tag{25}$$

The components of the state vector $x \to R^4$ and the control vector $u \to R^2$ are given by the following:

$x_1 \to$ horizontal velocity, kt
$x_2 \to$ vertical velocity, kt
$x_3 \to$ pitch rate, deg/s
$x_4 \to$ pitch angle, deg
$u_1 \to$ "collective" pitch control
$u_2 \to$ "longitudinal cyclic" pitch control

Essentially, control is achieved by varying the angle of attack with respect to the air of the rotor blades. The collective control u_1 is mainly used for controlling the motion of the aircraft vertically up and down. Control u_2 is used to control the horizontal velocity of the aircraft.

In Ref. 12, the linearized mathematical model is presented assuming a nominal airspeed to be 135 kt. It is also shown that during operation in the flight envelope of interest significant changes take place only in the elements a_{32}, a_{34}, and b_{21}. The ranges of values taken by these elements are given by

$$0.0663 \le \bar{a}_{32}(\simeq 0.3681) \le 0.5044$$

$$0.1220 \le \bar{a}_{34}(\simeq 1.4220) \le 2.5280$$

where $(\bar{\ })$ denotes the nominal value.

Note that the perturbation ranges are asymmetric with respect to the nominal values. To take full advantage of the perturbation bound analysis, we will "bias" the nominal values of a_{32}, a_{34}, and b_{21} such that we obtain the symmetric ranges. Accordingly, the nominal values of a_{32}, a_{34}, and b_{21} now are $\bar{a}_{32} = 0.2855$, $\bar{a}_{34} = 1.3229$, and $\bar{b}_{21} = 3.04475$. The full matrices A and B are given by

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.4422 & 3.0447 & -5.5200 & 0.0000 \\ 0.1761 & -7.5922 & 4.4900 & 0.0000 \end{bmatrix}$$

so that

$$|\Delta A_{32}|_{\max} = 0.2197$$

$$|\Delta A_{34}|_{\max} = 1.2031$$

$$|\Delta B_{21}|_{\max} = 2.06725$$

In Yedavalli and Liang,[13] a robust constant gain linear state feedback control law that stabilized the system in the entire range of the perturbation was obtained that is given by

$$G = \begin{bmatrix} -0.4670 & 0.0139 & 0.5390 & 0.8060 \\ 0.0430 & 0.5190 & -0.1899 & -0.7310 \end{bmatrix}$$

and the nominal closed-loop system matrix is given by

$$A + BG = \begin{bmatrix} -0.2355 & 0.1246 & 0.2237 & -0.2278 \\ -1.7002 & -4.9080 & 3.0853 & 3.9832 \\ 2.8711 & 2.5392 & -4.5349 & -6.4084 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

However, the previous control gain was shown to be a robust control gain by using Lyapunov-based stability robustness analysis, which involved the use of similarity transformation and other problem-specific techniques. We now use the proposed new stability robustness bounds to show that the previous gain is indeed a robustly stabilizing gain.

Since the range of ΔA_{32}, ΔA_{34}, and ΔB_{21} are known, the corresponding perturbation structure matrices A_i of Eq. (17) are given by

$$A_1 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.1063 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5820 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.4670 & 0.0139 & 0.5390 & 0.8060 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

With the previous matrices the computation of the proposed bounds yields

$$\mu_k = 2.5235$$

$$\mu_L = 2.9421$$

$$\mu_G = 3.7397$$

Since these bounds exceed $|\Delta B_{21}|_{\max} = 2.06725$, we conclude that the previous gain indeed stabilizes the system in the entire range of the parameters.

Application to Drone Lateral Attitude Control

The system matrices for the drone lateral attitude control system considered in Ref. 14 are given by

$$A = \begin{bmatrix} -0.0853 & -0.0001 & -0.9994 & 0.0414 & 0.0000 & 0.1862 \\ -46.8600 & -2.7570 & 0.3896 & 0.0000 & -124.3000 & 128.6000 \\ -0.4248 & -0.0622 & -0.0671 & 0.0000 & -8.7920 & -20.4600 \\ 0.0000 & 1.0000 & 0.0523 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 20.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 20.0 \end{bmatrix}$$

With a linear state feedback control gain

$$G = \begin{bmatrix} -215.1000 & 4.6650 & 7.8950 & 233.2000 & -6.7080 & 2.5540 \\ -231.5000 & -3.7230 & 7.4530 & -213.5000 & 2.5540 & -6.8690 \end{bmatrix}$$

the closed-loop system matrix $\bar{A} = A + BG$ is made asymptotically stable.

Now assuming the element A_{21} to be the uncertain parameter (having a nominal value $= -46.86$), we get the stability robustness bound on this parameter as

$$\mu_{21_k} = 8.9458e + 03$$

$$\mu_{21_L} = 9.7711e + 03$$

$$\mu_{21_G} = 2.3472e + 04$$

For the previous problem the bound using Lyapunov theory that was given in Ref. 15 is shown to be

$$\mu_{21} = 573.46$$

Thus, it is clear that the proposed bounds that use Kronecker theory are much less conservative compared with the bounds derived using Lyapunov theory.

Remark 1: It is interesting to note that, in all of the previous examples, the bound based on the bialternate sum matrix gives the best bound. This has been the case for the majority of the other problems that were worked out. It is important to realize that the proposed bounds are calculated directly in matrix domain without resorting to "frequency sweeping" (as is done in Refs. 4 and 5) and "parameter gridding."

Some preliminary and partial results of this paper were reported in Yedavalli.[16]

V. Conclusions

The paper presents new, much less conservative sufficient bounds for robust stability of linear state-space models. The technique involves building large order matrices but is suitable for analyzing the robust stability for a large number of uncertain parameters.

The results of this paper can be used to design controllers for stability robustness as well as for gain scheduling purposes.

Appendix A: Proof of Theorem 5

Necessity: If $A(q)$ is stable, then $K[A(q)]$ has negative real part eigenvalues. Hence, we have $(-1)^k \det K[A(q)]>0$; i.e., $(-1)^k \det\{K[A_0+E]\}>0 \Rightarrow (-1)^k\{\det[K[A_0]+(\Sigma f_i K \times [A_i])]\}>0$ (because $K[A+B]=K[A]+K[B]$).

Since $K[A_0]$ is stable, we can write

$$\det\left\{K[A_0]+\sum f_i K[A_i]\right\} = \det\left(K[A_0]\right)\cdot\det\left\{I_k+\left(K[A_0]\right)^{-1}\sum f_i K[A_i]\right\}$$

$$\det\left\{K[A_0]+\sum f_i K[A_i]\right\} = \det\left(K[A_0]\right)\cdot\det\left\{I_k+\sum f_i K[A_i]\left(K[A_0]\right)^{-1}\right\}$$

and noting that $K[A_0]$ is stable, and since $\det K[A_0]>0$, we can conclude that Eq. (19) is necessary.

Sufficiency:

$$\det\left\{I_k+\left(\sum f_i K[A_i]\right)\left(K[A_0]\right)^{-1}\right\}>0$$

and the fact that $K[A_0]$ is stable implies that

$$(-1)^k\left\{\det\left(K[A_0]\right)\cdot\det\left[I_k+\left(\sum f_i K[A_i]\right)\left(K[A_0]\right)^{-1}\right]\right\}>0$$

$$\Rightarrow (-1)^k \det K[A(q)]>0$$

$$\Rightarrow (-1)^k \Pi_i \lambda_{k_i}>0$$

where λ_{k_i} are the eigenvalues of $K[A(q)]$. Since λ_{k_i} are sums of eigenvalues λ_i of $A(q)$, it implies that λ_i cannot have zero real parts. But λ_i cannot have positive real parts either because A_0 is stable and $A(q)$ is a continuous function of the parameter vector q, which in turn implies that $A(q)$ is stable.

Appendix B: Proof of Theorem 8

Let

$$K[A_0]=B_0; \qquad K[A_i]=B_i; \qquad \max_q f_i(q-q^0)=\beta_i$$

From Theorem 5, it is known that a necessary and sufficient condition for stability of $A_0+E(q)$ is

$$\det\left[I+\left(\sum_i \beta_i B_i\right)B_0^{-1}\right]>0$$

This is satisfied if

$$\rho\left[\left(\sum \beta_i B_i\right)B_0^{-1}\right]<1$$

But

$$\rho\left[\left(\sum \beta_i B_i\right)B_0^{-1}\right]=\rho\left(\sum \beta_i B_i B_0^{-1}\right)\le\rho\left(\sum|\beta_i B_i B_0^{-1}|\right)$$

$$\le \max_j|\beta_j|\,\rho\left(\sum|B_i B_0^{-1}|\right)$$

Hence $A_0+E(q)$ is stable if

$$|\beta_i|<\frac{1}{\rho\left(\sum_{i=1}^{r}|B_i B_0^{-1}|\right)}$$

Acknowledgment

This work was supported by NASA Langley Research Center under Grant NAG-1-1164.

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